Pinning of vortices in a Bose-Einstein condensate by an optical lattice

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We consider the ground state of vortices in a Bose-Einstein condensate. We show that turning on a weak optical periodic potential leads to a transition from the triangular Abrikosov vortex lattice to phases where the vortices are pinned by the optical potential. We discuss the phase diagram of the system for a two-dimensional optical periodic potential with one vortex per optical lattice cell. We also discuss the influence of a one-dimensional optical periodic potential on the vortex ground state. The latter situation has no analogue in other condensed-matter systems.

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Introduction — The effects of a periodic array of pinning centers on vortices in superconducting materials have attracted a lot of experimental [1, 2, 3, 4, 5] and theoretical [6, 7, 8, 9, 10, 11, 12, 13] attention. Of particular interest is the effect of the pinning potential on the melting of a vortex lattice. The vortex lattice is known to melt via a first-order transition in clean materials [14], whereas the presence of pinning centers significantly enriches the phase diagram, due to the intricate interplay between the vortex-vortex interactions, pinning potential, and thermal and quantum fluctuations [6, 7, 8]. At zero temperature and for strong pinning, the system has, depending on the number of vortices per pinning center, i.e., the filling factor, various phases where the vortices order in a periodic array [9, 10, 11, 12]. If the pinning potential is weakened, the pinned vortex lattice undergoes a first-order transition to a deformed triangular Abrikosov lattice [13].

Recently, the experimental study of vortices in superfluids has been complemented by the experiments with rotating atomic Bose gases [15, 16, 17, 18]. Within this field it has become possible to experimentally study the dynamics of a single vortex line in great detail [19, 20], leading to an enhancement of the theoretical interest in the dynamics of a single vortex in a Bose-Einstein condensed atomic gas [21].

Another interesting development in the field of atomic gases is the possibility to trap atoms in a periodic potential using a so-called optical lattice. Here, one uses the dipole force which the atoms experience in an off-resonant light field. Using an optical lattice Greiner *et al.* [22] were able to experimentally observe the transition from a superfluid, where the atoms are delocalized across the lattice, to a Mott-insulating state where the atoms are localized onsite [23].

A common feature of the experiments with ultracold atomic gases is that these systems are very clean. Therefore, vortices in a Bose-Einstein condensate do not experience an intrinsic pinning potential and the observed vortex lattices are triangular Abrikosov lattices. In this Letter, we study the ground state of vortices in a Bose-Einstein condensate. We show that turning on an optical periodic potential leads to a transition from the triangular Abrikosov lattice to phases where the vortices are pinned by the optical potential. We restrict ourselves to the case with one vortex per unit cell of the two dimensional optical lattice.

Interestingly, the precise knowledge of the optical lattice potential allows for a microscopic and quantitative calculation of the phase diagram, as opposed to superconducting materials, where the pinning potential is known only phenomenologically. In the experiments with rotating Bose-Einstein condensates, the vortex lattices are relatively easy to observe, which allows for a detailed experimental study of the transitions between the various phases of the vortex lattice as one tunes the strength of the optical potential. Moreover, applying a one-dimensional optical lattice leads to pinning "valleys" instead of pinning centers, and, therefore, to pinned phases of the vortex lattice which have no analogue in other condensed-matter systems.

Two-dimensional optical lattice — Our starting point is the hamiltonian functional for the macroscopic condensate wave function $\Psi(\mathbf{x})$, given by

$$H[\Psi^*,\Psi] = \int d\mathbf{x}\Psi^*(\mathbf{x})\left[-\frac{\hbar^2\nabla^2}{2m}+\frac{1}{2}g|\Psi(\mathbf{x})|^2 + V_{\mathrm{OL}}(\mathbf{x})-\mu\right]\Psi(\mathbf{x}) . \qquad (1)$$

Here, m denotes the mass of one atom which interacts with the other atoms via a two-body contact interaction of strength $g=4\pi a_s\hbar^2/m$, with a_s the s-wave scattering length. The optical lattice potential is given by $V_{\mathrm{OL}}(\mathbf{x}) = sE_{\mathrm{R}}[\sin^2(qx)+\sin^2(qy)]$ with E_{R} the recoil

energy, q the wavenumber of the lattice, and $s \geq 0$ a dimensionless number indicating the strength of the optical lattice. Note that considering one vortex per unit cell of the optical lattice implies that we take the two-dimensional vortex density equal to $n_{\rm v} = q^2/\pi^2$. The chemical potential that fixes the number of atoms in the condensate is represented by μ.

In the following we consider for simplicity a condensate with infinite extent in the x-y-plane and tightly confined in the z-direction by an harmonic trap with frequency ω_z. The latter assumption allows us to neglect the curvature of the vortex lines, and leads effectively to a condensate thickness $d \equiv \sqrt{\pi\hbar/(m\omega_z)}$ in the z-direction. In the Thomas-Fermi limit, where we neglect the kinetic energy of the condensate atoms with respect to their mean-field interaction energy, the global density profile of the condensate in the optical lattice is given by $n_{\rm TF}(\mathbf{x}) = n - [V_{\rm OL}(\mathbf{x}) - sE_{\rm R}]/g$, with $n = [\mu - sE_{\rm R}]/g$ the average density of the condensate. To find the potential energy of a vortex in a Bose-Einstein condensate in an optical lattice as function of its coordinates (u_x, u_y), we use the variational *ansatz* for the condensate wave function

$$\Psi(\mathbf{x}) = \sqrt{n_{\rm TF}(\mathbf{x})}\ \Theta\left[|\mathbf{x}-\mathbf{u}|/\xi - 1\right] \exp[i\phi(\mathbf{x},\mathbf{u})]\ , \quad (2)$$

with $\xi = 1/\sqrt{8\pi a_s n}$ the healing length that sets the size of the vortex core, $\phi(\mathbf{x},\mathbf{u}) = \arctan[(y-u_y)/(x-u_x)]$ the phase of the vortex, and $\Theta(z)$ the unit step function. For the above *ansatz* to be valid, we have assumed that the vortex core is much smaller then an optical lattice period, $q\xi \ll 1$, and that the strength of the potential is sufficiently weak, $sE_{\rm R} \ll \mu$. Note that the above variational *ansatz* indeed describes a vortex along the z-axis at position (u_x, u_y) in the x-y-plane.

Substitution of the *ansatz* in Eq. (2) in the hamiltonian functional in Eq. (1), and isolating the contribution due to the presence of the vortex leads to [24]

$$U_{\rm 2D}(\mathbf{u}) = \frac{d}{8a_s} sE_{\rm R} Q(q\xi)[\cos(2qu_x) + \cos(2qu_y)]\ . \quad (3)$$

Here, we defined $Q(z) = J_1(2z)/(2z) + \int_1^\infty d\rho J_0(2z\rho)/\rho$, with J_0 and J_1 the zeroth and first order Bessel function of the first kind. It is clearly seen that the potential energy is minimal if the vortices are located at the maxima of the optical potential. This is expected, since at these maxima the condensate density, and hence the kinetic energy associated with the superfluid motion, is minimal. The expression in Eq. (3) is the pinning potential experienced by vortices in a condensate loaded in a optical lattice. If the pinning potential is sufficiently strong and we have one vortex per unit cell of the optical lattice, the ground state is a configuration in which each vortex is trapped or pinned by an optical lattice maximum. For a two-dimensional optical periodic potential, this is the square pinned lattice (SP), shown in Fig. 1.

To determine the phase diagram in detail, we calculate the total energy per vortex for an arbitrary vortex lattice. This approach neglects the fact that for very weak pinning potentials the triangular Abrikosov lattice will be slightly deformed [13]. Generally, the vortex lattice is parametrized as follows

$$\mathbf{u}(\alpha,\beta) = \frac{\pi}{q}\begin{pmatrix} \sqrt{1+\alpha\beta} & \alpha \\ \beta & \sqrt{1+\alpha\beta} \end{pmatrix}\begin{pmatrix} n_x \\ n_y \end{pmatrix}\ , \quad (4)$$

with $n_i \in \mathbb{Z}$ and $0 \leq \alpha, \beta \leq \frac{1}{2}$. The transformation matrix conserves the area of the unit cell, and thus ensures that we are considering configurations with equal vortex density. The more familiar parameters of a unit cell of a two-dimensional lattice, the angle φ and the ratio of the length of the sides of the cell $\kappa = L_1/L_2$, are related to α and β by

$$\frac{\cos\varphi}{\kappa} = \frac{(\alpha+\beta)\sqrt{1+\alpha\beta}}{1+\alpha\beta+\alpha^2}, \quad \frac{\sin\varphi}{\kappa} = \frac{1}{1+\alpha\beta+\alpha^2}. \quad (5)$$

The pinning energy per vortex is found by substituting Eq. (4) in Eq. (3), summing over all n_i, and dividing the result by the number of unit cells. In the limit $n_i \to \infty$ we find

$$E_{\rm pin}^{\rm 2D}(\alpha,\beta) = -\frac{d}{8a_s} sE_{\rm R} Q(q\xi)[\delta_{\beta,0} + \delta_{\alpha,0}]\ . \quad (6)$$

Our next task is to determine the interaction energy per vortex. In two dimensions, singly-quantized vortices with equal orientation experience a logarithmic long-range interaction $V(r) = -2\pi d\hbar^2 n/m \log(r/\xi)$ for $r \gg \xi$ [25]. The interaction energy $E_{\rm int}$ per vortex for an infinite two-dimensional lattice of vortices was calculated by Campbell *et al.* [26]. Cast in a dimensionless form, their result reads

$$\begin{aligned} \tilde{E}_{\rm int} \equiv \frac{E_{\rm int}}{(\pi\hbar^2 dn/m)} &= \frac{\pi}{6}\frac{\sin\varphi}{\kappa} - \log\left[2\pi\left(\frac{\sin\varphi}{\kappa}\right)^{\frac{1}{2}}\right] \\ &- \log\left\{\Pi_{j=1}^{\infty}[1 - 2e^{-2\pi j|\sin\varphi|/\kappa}\cos\left(2\pi j\frac{\cos\varphi}{\kappa}\right)\right. \\ &\left. + e^{-4\pi j|\sin\varphi|/\kappa}]\right\}\ . \end{aligned} \quad (7)$$

It is important to realize that the interaction energy per vortex is divergent for an infinite lattice, and that the above expression gives the relative interaction energy for configurations with equal vortex density.

The expression for $E_{\rm int}$ together with the expression for the pinning energy in Eq. (6) enables us to minimize the total energy as a function of α and β, and to determine the vortex lattice ground state for different strengths of the periodical optical potential. Performing this procedure as function of the dimensionless variables $q\xi$ and $sE_{\rm R}/\mu$ leads to three phases, which are physically due to the the competition between pinning and interactions. For a very weak optical potential we find that vortices arrange themselves on a triangular Abrikosov lattice (AB), with $\alpha = \beta = \sqrt{1/\sqrt{3} - 1/2}$, as expected. In

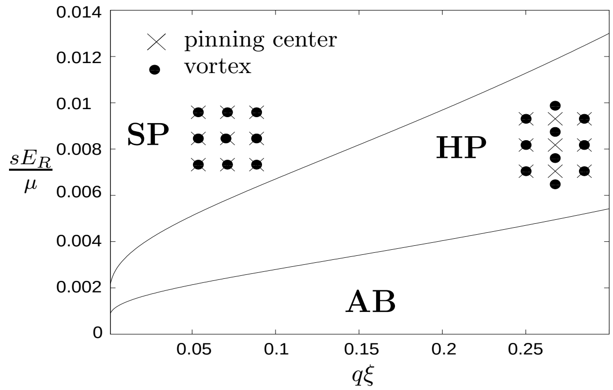

FIG. 1: Vortex phase diagram of a Bose-Einstein condensate in a two-dimensional optical square lattice, with one vortex per unit cell of the optical lattice. Three phases are relevant: a square and fully pinned configuration (SP), a triangular configuration where half of the vortices are located at the pinning centers (HP), and the unpinned triangular Abrikosov vortex lattice (AB).

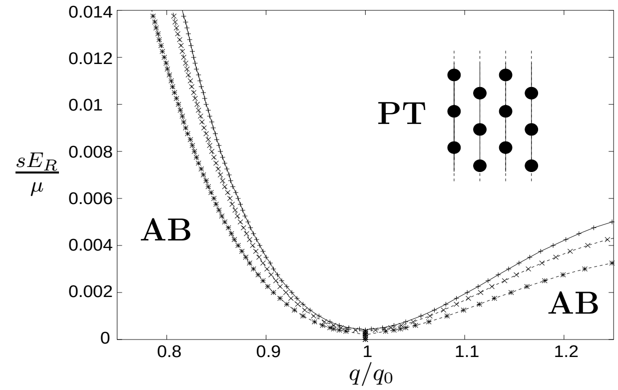

FIG. 2: Vortex phase diagram of a Bose-Einstein condensate in a one-dimensional optical lattice. Two phases are relevant: a pinned triangular configuration (PT), and the unpinned triangular Abrikosov vortex lattice (AB). We calculated the phase boundaries for $\xi = .01\pi/q_0$ (+), $\xi = .005\pi/q_0$ ($\times$) and $\xi = .001\pi/q_0$ ($*$). Note that at the line $q/q_0 = 1$ the phases coincide.

this phase, with $\tilde{E}_{\rm int} = -1.32112$, the interactions dominate over the pinning. On the other hand, when the pinning energy dominates over the interaction, we find the square pinned lattice ($\alpha = \beta = 0$) [13]. This phase has $\tilde{E}_{\rm int} = -1.31053$. In the intermediate regime, where the pinning and interactions are equally important, we find a phase in which half of the vortices are pinned (HP) [13], and the lattice has a triangular structure ($\alpha = \frac{1}{2}$, $\beta = 0$ and $\tilde{E}_{\rm int} = -1.31800$). In the zero-temperature phase diagram, shown in Fig. 1, the different phases are separated by first-order phase transitions and the phase boundaries are given by

$$\left(\frac{sE_{\rm R}}{\mu}\right)_{\rm AB-HP} = \frac{.00623}{Q(q\xi)}, \quad \left(\frac{sE_{\rm R}}{\mu}\right)_{\rm HP-SP} = \frac{.01494}{Q(q\xi)}. \tag{8}$$

One-dimensional optical lattice — For a one-dimensional optical lattice the single vortex pinning potential equals $U_{\rm 1D}(\mathbf{u}) = \frac{d}{8a_s} sE_{\rm R} Q(q\xi)\cos(2qu_x)$ and hence the minima of $U_{\rm 1D}$ act as pinning "valleys". The pinning energy per vortex reads $E_{\rm pin}^{\rm 1D} = -\frac{d}{8a_s} sE_{\rm R} Q(q\xi)\delta_{\beta,0}$ within the parametrization in Eq. (4). Two phases are distinguished in this case, i.e., a pinned triangular lattice (PT), shown in Fig. 2, and the unpinned Abrikosov vortex lattice. Interestingly, the interactions always favor a triangular lattice since the vortices are allowed to arrange freely in the y-direction.

Consider now the case that the wavenumber of the optical lattice is such that the AB lattice and the PT lattice coincide, i.e., $q_0 \equiv 2\pi\sqrt{3}/(3L)$, with L the intervortex distance. If this configuration is disturbed by changing the optical lattice wavenumber to arbitrary q there will be a competition between the AB lattice and the PT lattice. The unit cell of the latter is, for wavenumber q and at equal vortex density, described by

$$\frac{\cos\varphi}{\kappa} = \frac{2}{1+3\left(\frac{q_0}{q}\right)^4}, \quad \frac{\sin\varphi}{\kappa} = \frac{2\sqrt{3}\left(\frac{q_0}{q}\right)^2}{1+3\left(\frac{q_0}{q}\right)^4}. \tag{9}$$

The interaction energy per unit cell is found by substitution of Eq. (9) in Eq. (7). From this we find the zero-temperature phase diagram, depicted in Fig. 2 for various values of the healing length. The generic behavior is such that for given strength of the optical lattice and for small deviations from q_0, the vortex lattice stays pinned, i.e., the vortices are dragged along with the pinning valleys. At certain q the phase boundary is crossed. Then the vortices "jump" back to their original positions, forming an Abrikosov lattice again.

Discussion and conclusions — Since we have considered the ground state of vortices, we have implicitly assumed that the optical lattice is co-rotating with the Bose-Einstein condensate. Although this is very difficult to achieve experimentally, it has, however, recently been proposed to create effective magnetic fields, and therefore effective rotation, by optical methods [27, 28]. We believe that the calculations presented here are relevant for such a situation.

Of particular experimental interest are the collective modes supported by the pinned vortex lattices. In the absence of an optical lattice potential, the dispersion relation for the gapless Tkachenko modes has been measured by Coddington *et al.* [29]. We expect that these modes acquire a gap in the presence of a periodic optical potential due to the fact that the translational symmetry is not broken spontaneously, but by the optical lattice. In the case of the SP lattice this gap is easy to calculate, since the zero-momentum Tkachenko mode corresponds

in this case to a simultaneous in-phase precession of all the vortices around the maxima of the optical lattice. Hence the gap is given by $\hbar\omega_{\rm p}$, with $\omega_{\rm p}$ the precession frequency [21]

$$\omega_{\rm p} = \frac{\pi\hbar q^2 Q(q\xi) s E_{\rm R}}{mgn} \ . \tag{10}$$

In future work we intend to study the collective modes of the pinned vortex lattices in great detail.

In this Letter we have restriced ourselves to the case of one vortex per unit cell of the optical lattice. Another direction for future work will be a detailed study of the pinned phases at other filling factors, where other types of pinned vortex lattices are known to occur [9, 10, 11, 12]. Since we have only been considering infinite lattices, we intend to study also the effects of the finite size of the system, which may be significant [30]. With respect to this latter remark it is also important to note that the harmonic magnetic trap used in the experiments induces an additional feature in the density profile of the condensate that may have important effects [31], and therefore also requires further study.

Apart from these interesting possibilities, yet another direction would be to consider more strongly-correlated regimes that occur at fast rotation, and to study the effects of the periodic optical potential on the melting of the vortex lattice [32]. One would expect that in this regime the effect of quantum fluctuations, i.e., quantum tunneling of the vortices through the potential barriers of the pinning potential, becomes important. Conversely, in the limit of a strong optical periodic potential, an interesting topic is to study the effects of the rotation on the Mott-Insulator transition [33]. In conclusion, rotating Bose gases in an optical lattice provide an interesting system to study new quantum phases of matter [34], as well as phenomena known from condensed matter in a new context.

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